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The Decomposition of $C_r(K)$ into the Direct Sum of Subalgebras

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Let A and B be closed subalgebras of $C_r(X)$ whose direct sum is $C_r(X)$. Some consequences of this relation are explored in this paper. For example if $1 \in A$ (as may be assumed) it is shown that the norm of the projection onto A is an odd integer and there is a retraction of X onto the set of common zeros of elements of B.

INTRODUCTION

Let K be a compact Hausdorff space and $C_r(K)$ the algebra of continuous real-valued functions on K. If A and B are two closed subspaces of $C_r(K)$ with only zero in common and if each element $h \in C_r(K)$ has the form $h = f + g$ where $f \in A$ and $g \in B$ then we write

$$C_r(K) = A \oplus B$$

and say $C_r(K)$ is the direct sum of A and B. This can happen in many ways. This paper examines what follows if we assume in addition that both A and B are algebras under the usual pointwise multiplication. It turns out that there are some rather surprising consequences of this assumption which are detailed in the theorem and its corollary. We begin the paper with two examples which will help to orient the reader to the variety of possibilities that may occur. The main results follow the examples and the paper concludes with some further examples which delineate the applicability of the theorem and illustrate its conclusions.

EXAMPLE 1. Let τ be a continuous function from K into K which is the identity on its range; i.e., $\tau(\tau(x)) = \tau(x)$ for all $x \in K$. Let

$$A = \{f \in C_r(K): f(\tau(x)) = f(x) \text{ for all } x \in K\}$$

and

$$B = \{g \in C_r(K): g(\tau(x)) = 0 \text{ for all } x \in K\}.$$

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Then A and B are closed subalgebras and $A \cap B = \{0\}$. If $h \in C_r(K)$, then $h \circ \tau$ is in A while $h - h \circ \tau$ is in B. This shows that the direct sum of A and B is $C_r(K)$. Note that the norm of the projection of $C_r(K)$ onto A is one. Also note that the case when B is a maximal ideal and A the real numbers corresponds exactly to the case when τ is a constant.

EXAMPLE 2. Let m be a positive integer and let $K = [0,2m + 1]$. Let A consist of those functions $f \in C_r(K)$ with

(i) f constant on each interval $[2j, 2j + 1]$, $j = 0,\dots, m$

and let B consist of those functions $g \in C_r(K)$ with

(ii) $g(0) = (0)$

(iii) g constant on each interval $[2j + 1, 2j + 2]$, $j = 0,\dots, m - 1$.

Then A and B are closed subalgebras of $C_r(K)$, $1 \in A$, and $A \cap B = \{0\}$. Further, if $h \in C_r(K)$ define $f \in A$ and $g \in B$ by

$$f(x) = \sum_0^{2k} (-1)^j h(j), \qquad 2k \leqslant x \leqslant 2k + 1, \quad k = 0,\dots, m$$

and

$$g(x) = \sum_0^{2k+1} (-1)^{j+1} h(j), \qquad 2k + 1 \leqslant x \leqslant 2k + 2, \quad k = 0,\dots, m - 1$$

and

$$f(x) = h(x) - g(x), \qquad 2k + 1 \leqslant x \leqslant 2k + 2$$

$$g(x) = h(x) - f(x), \qquad 2k \leqslant x \leqslant 2k + 1.$$

Clearly, $f + g = h$ so that $A \oplus B = C_r(K)$. Finally, note that if π_A is the projection of $C_r(K)$ onto A, then $\| \pi_A \| = 2m + 1$. As we will see in the theorem, Example 2 is fairly typical.

PROPOSITION. *Suppose $C_r(K) = A \oplus B$ where A and B are closed subalgebras. Then there is an open-closed set L in K with characteristic function χ such that $\chi \in A$, $1 - \chi \in B$ and restrictions of both A and B to L and $K \backslash L$ are closed.*

Proof. Let $1 = a_0 + b_0$ where $a_0 \in A$, $b_0 \in B$. If $f \in A$, then $f = a_0 f + b_0 f$ so that $b_0 A \subseteq A$. Likewise, $a_0 B \subseteq B$. Hence, $a_0 b_0 \in A \cap B = \{0\}$ so that

$$1 = 1^2 = (a_0 + b_0)^2 = a_0^2 + b_0^2$$

Thus, $a_0^2 = a_0$, $b_0^2 = b_0$ and so $a_0 = \chi$, $b_0 = 1 - \chi$. Next, suppose $f_n \in A$ and $f_n \to f \in C_r(K)$ uniformly on $L = \operatorname{supp}(\chi)$. Then $a_0 f_n \to a_0 f$ uniformly on K so that $a_0 f \in A$ and the restriction of f to L belongs to the restriction of A to L. The other assertions follow in a like manner.

The proposition shows that there is no loss of generality in assuming that A contains the constants.

THEOREM. *Let $C_r(K) = A \oplus B$ where A and B are closed proper subalgebras and $1 \in A$. Then the set Z of common zeros of elements of B is non-empty and there is a retraction of K onto Z. Further, if π_A is the projection of $C_r(K)$ onto A, then the norm of π_A is an odd integer; if π_A has norm 1 then there is a continuous function $\tau: K \to K$ with $\tau \circ \tau = \tau$ and $A = \{f: f \circ \tau = f\}$, $B = \{g: g \circ \tau = 0\}$.*

Proof. We first show that Z is non-empty. Let $c \in \mathbb{R}$ and suppose that $\operatorname{dist}(c, B) < 1$; let $v \in B$ satisfy $\| c + v \|_\infty < 1$. Then $\|(c + v)^n\| = \| c + v \|^n < 1$ for $n = 1, 2, \ldots$ and $\pi_A((c + v)^n) = c^n$ since B is an angebra. Hence, $| c |^n \leqslant \| \pi_A \|$ so that $| c | \leqslant 1$. Thus, $| a | = \operatorname{dist}(a, B)$ for all real numbers a. It follows directly by the Hahn-Banach theorem that there is a real measure μ on K of mass one which is orthogonal to B and $\int 1 \, d\mu = \operatorname{dist}(1, B) = 1$. Hence, μ is non-negative. Since $| g | \in B$ whenever $g \in B$ we have $0 = \int | g | \, d\mu$ for all $g \in B$. Consequently, the closed support of μ lies in the set of common zeros of the elements of B and, in particular, Z is non-empty.

If x and y are two points of K we shall say that x and y are A-related if $f(x) = f(y)$ for all $f \in A$; define B-related similarly. Let Q_A be the function which assigns to each $x \in K$ its A-equivalence class and let Q_B be the similar function for B. If we give $K_A = Q_A(K)$ the quotient topology, then each continuous real-valued function F on K_A is of the form $F(Q_A(x)) = f(x)$ for some $f \in A$ by the Stone-Weierstrass theorem. Likewise, if $K_B = Q_B(K)$ is given the quotient topology and if $\xi = Q_B(Z)$, then each continuous real-valued function G on K_B which vanishes at ξ is of the form $G(Q_B(x)) = g(x)$ for some $g \in B$.

For $x \in K$ let λ_x be the unique measure determined by

$$\int_K h \, d\lambda_x = (\pi_A h)(x), \qquad h \in C_r(K).$$

We shall show that λ_x is the combination of $2s + 1$ point masses with weights one or minus one where, of course, $2s + 1 \leqslant \| \pi_A \|$. Since

$$\sup_{x \in K} \| \lambda_x \| = \| \pi_A \|$$

this will show that $\| \pi_A \|$ is an odd integer.

Let $\mathcal{O}$ be an open set in K which does not meet Z. I claim that there must be a point in $\mathcal{O}$ which is either A- or B-related to some point not in $\mathcal{O}$. Suppose not. Let $x \in \mathcal{O}$ and let $W = \cup_j K_j$ where each K_j is compact, the K_j increase, lie in $K \backslash Z$, and λ_x has no mass on $\mathcal{O} \backslash W$. Let $X = K \backslash \mathcal{O}$. Then $Q_A(K_j) \cap Q_A(X) = \varnothing$; since A is the pull-back of $C_r(K_A)$ there is an $F_j \in A$ with $F_j = 1$ on K_j, $F_j = 0$ on X, and $0 \leqslant F_j \leqslant 1$ on K. Likewise, there is a $G_j \in B$ with $G_j = 1$ on K_j, $G_j = 0$ on X, and $0 \leqslant G_j \leqslant 1$ on K. Thus, both the sequences $\{F_j\}$ and $\{G_j\}$

converge boundedly and pointwise a.e. λ_x on K to the characteristic function of $\mathcal{O}$. Hence, we have both

$$1 = \lim_j \int F_j \, d\lambda_x = \lambda_x(\mathcal{O})$$

and

$$0 = \lim_j \int G_j \, d\lambda_x = \lambda_x(\mathcal{O}),$$

an obvious contradiction. (We assume with no loss that $x \in K_j$ for all j.)

Taking $\mathcal{O} = K \backslash Z$ in the above we see that there is at least one point $p \notin Z$ which is A-related to some point $z \in Z$.

In what follows, δ_p is the point mass at $p \in K$. Suppose $x \in K$ and x is A-related to some $z \in Z$. Then $\lambda_x - \delta_z$ annihilates both A and B and so is zero. Thus, $\lambda_x = \delta_z$. Next, suppose $x \in K$ and x is B-related to some y which is in turn A-related to some $z \in Z$. Then $\lambda_x - \delta_x + \delta_y - \delta_z$ is orthogonal to both A and B and so $\lambda_x = \delta_x - \delta_y + \delta_z$. We can continue this process: for $r \geqslant 1$ we shall say that $x \in K$ is r-linked to Z if there are r points $x_1, \ldots, x_r$ with x_1 A-related to some $z \in Z$, x_2 B-related to x_1, etc., terminating with x either A- or B-related to x_r depending on whether r is even or odd, respectively; we define the points of Z to be zero-linked to Z. Note that if x is r-linked to Z and y is, for example, A-related to x, then either y is $r + 1$-linked to Z (for r odd) or r-linked to Z (for r even) since "A-related" is an equivalence relation and, in particular, is transitive. If $r = 2s$, then as above we see that

$$\lambda_x = \sum_{j=1}^{s} (\delta_{x_{2j}} - \delta_{x_{2j-1}}) + \delta_z . \tag{1}$$

If $r = 2s + 1$, then

$$\lambda_x = \delta_x - \delta_{x_r} + \sum_{j=1}^{s} (\delta_{x_{2j}} - \delta_{x_{2j-1}}) + \delta_z \tag{2}$$

Since $\| \lambda_x \| \leqslant \| \pi_A \|$ we find that if a point x can be r-linked to Z for some (possibly large) integer r then x can be r-linked to Z for an r which is less than $\| \pi_A \|$; that is, in formulas (1) and (2) there can be at most $\| \pi_A \|$ distinct points. It follows that the set of points which can be r-linked to Z for some r is closed. If we let $\mathcal{O}$ be its complement then the paragraph above implies that $\mathcal{O}$ must be empty for by definition no point of $\mathcal{O}$ can be either A or B related to any point not in $\mathcal{O}$. Hence, each point of K is r-linked to Z and λ_x has the form (1) or (2) for distinct points $x, x_1, \ldots, x_r, z$ with $z \in Z$ and no other x_j in Z. Note if x is already in Z, then $\lambda_x = \delta_x$. This proves that $\| \pi_A \|$ is an odd integer.

Now let τ be the function on K given by $\tau(x) = z$ where z is related to x as in (1) or (2). τ is then the identity on Z and τ maps X into Z. Note τ is constant on each A-equivalence class since $\lambda_x = \lambda_y$ if x is A-related to y. If $x_n \to x$ in K, then

$$\lambda_{x_n} = \sum_j (\delta_{p_j^n} - \delta_{q_j^n}) + \delta_{z^n}$$

where p_j^n is B-related to q_j^n, and $z^n \in Z$. By compactness we may assume that $p_j^n \to p_j$, $q_j^n \to q_j$, and $z^n \to z \in Z$. Thus,

$$\lambda_x = \sum_j (\delta_{p_j} - \delta_{q_j}) + \delta_z$$

and $\tau(x_n) \to z = \tau(x)$. This implies τ is continuous and hence τ is a retraction of K onto Z.

Lastly, if $\| \pi_A \| = 1$, then $\lambda_x = \delta_z$ for each $x \in K$ where x is A-related to z and $z \in Z$. Thus,

$$(\pi_A h)(x) = h(z), \qquad h \in C_r(K)$$

so that $f(\tau(x)) = f(x), f \in A$, and $g(\tau(x)) = 0, g \in B$.

COROLLARY. *Let $z_0 \in Z$ and let $W = \{x \in K: f(x) = f(z_0)$ for all $f \in A\}$. Then there is a retraction of K onto W.*

Proof. Let $A_0 = \{f \in A: f(z_0) = 0\}$ and $B_1 = B \oplus \mathbb{R}$. Then A_0 and B_1 are closed subalgebras, $1 \in B_1$, and $A_0 \oplus B_1 = C_r(K)$. Hence, the theorem implies there is a retraction of K onto the set of common zeros of A_0; this set coincides with W.

EXAMPLE 3. Let K be the unit square in the xy-plane and let A consist of all those elements of $C_r(K)$ which are constant on each vertical line in K; that is, $f(x, y) = f(x, 0)$ for $0 \leqslant y \leqslant 1$ and each x, $0 \leqslant x \leqslant 1$. Let B consist of those elements g of $C_r(K)$ which satisfy

$$\int_0^1 g(x, y)\, dy = 0, \qquad \text{all } x, \quad 0 \leqslant x \leqslant 1.$$

Then A and B are closed subspaces, $1 \in A$, and A is an algebra. Further, $A \cap B = \{0\}$. Finally, if $h \in C_r(K)$, let f be defined by

$$f(x, y) = \int_0^1 h(x, t)\, dt, \qquad 0 \leqslant x, \quad y \leqslant 1,$$

so that $f \in A$. If $g = h - f$, then $g \in B$ so that $C_r(K)$ is the direct sum of A and B. Note that $\| \pi_A \| = 1$ and there is no point in K at which all the elements of B vanish. This shows that both A and B must be algebras in order that the conclusions of the theorem hold.

EXAMPLE 4. Let Δ be the closed unit disc in the xy-plane and let T be the unit circle, $x^2 + y^2 = 1$. Let B consist of those elements g in $C_r(\Delta)$ with $g = 0$ on T. Then there is no closed algebra A with $A \oplus B = C_r(\Delta)$. For if such an A existed, we would have $1 \in A$ (since Δ is connected) and the theorem would imply a contradiction since a disc can not be retracted onto its boundary.

On the other hand, if we let A consist of those real functions u which are harmonic in int Δ and continuous on Δ, then the direct sum of A and B is $C_r(\Delta)$. Hence, B is a closed subalgebra of $C_r(\Delta)$ which is complemented by a subspace but not by a subalgebra.

Likewise, if A consists of those functions $f \in C_r(\Delta)$ with f constant on T and $f(T) = f(0)$, then A can not be complemented by an algebra B whose zero set contains either 0 or any point of T.